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Best Rational Starting Approximations and Improved Newton Iteration for the Square Root

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Abstract. The most important class of the best rational approximations to the square root is obtained analytically by means of elliptic function theory. An improvement of the Newton iteration procedure is proposed.

1. Introduction. The most efficient computing procedure for calculating $\sqrt{x}$ on an interval $[a, b]$ $(0 < a < b)$ is to apply the Newton iteration,

$$R_i = (R_{i-1} + x/R_{i-1})/2 = N(R_{i-1}) = N^i(R_0),$$

to an appropriate starting approximation R_0. The function commonly used for a starting approximation is a polynomial or a rational function of some prescribed degree which is the best with respect to a certain optimality criterion. Several writers have obtained various best starting approximations with respect to different criteria [2]–[7].

A seemingly reasonable criterion is *Chebyshev's*,

$$\max|R_0(x)/\sqrt{x} - 1| = \min,$$

but a more reasonable one is *Moursund's*,

$$\max|R_i(x)/\sqrt{x} - 1| = \min, \qquad i = 1, 2, \ldots,$$

since our purpose is to optimize the quality of R_i for some $i > 0$, not R_0 itself. Moursund [7] has pointed out that a function R_0 satisfying his criterion for $i = 1$ satisfies it for every $i > 1$. Another familiar criterion is, say, the *logarithmic criterion*,

$$\max|\log(R_0(x)/\sqrt{x})| = \min.$$

It is believed that this criterion has been used by many writers [2]–[5] for technical reasons to make the analysis simpler. Recently, Sterbenz and Fike [9], King and Phillips [8], and the present author [10], discovered independently a surprisingly simple relationship, Theorem 2 in Section 2, among the three criteria.

This paper presents two new contributions concerning the computation of the square root. The main contribution is a complete analytical theory for the most important class of the best rational approximations to the square root. As a matter of fact, it is a simple modification of the classical but practically unknown theory which Ahiezer [1] credits to Solotarev. On the basis of the theory, the tables of the best rational approximations in Moursund's sense are computed anew. The secondary

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contribution is an improvement of the Newton iteration for the square root itself, which leads to a striking acceleration of the convergence.

2. Preliminaries. In this section, various definitions and theorems concerning the characterization of, and the relation of the best rational approximations to the square root will be given for later references.

Let $\mathbf{R}(p, q)$ denote the set of rational functions of the form $R(x) = P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are mutually prime polynomials of degrees not exceeding p and q, respectively. For any $R \in \mathbf{R}(p, q)$ which is not identically 0, its *degree* $D(R)$ is defined by

$$D(R) = p + q - \min[p - p^*, q - q^*], \tag{1}$$

where p^* and q^* are the exact degrees of P and Q respectively.

A function $R^* \in \mathbf{R}(p, q)$ is called the *C-approximation*, the *M-approximation*, or the *L-approximation*, respectively, of the class (p, q) on $[a, b]$, if it satisfies

$$E_I(R^*) = \min[E_I(R); \quad R \in \mathbf{R}(p, q)], \qquad I = C, M, \text{or } L, \tag{2}$$

where

$$E_C(R) = \max[|R(x)/\sqrt{x} - 1|; \quad x \in [a, b]], \tag{3.1}$$

$$E_M(R) = \max[|(R(x) + x/R(x))/(2\sqrt{x}) - 1|; \quad x \in [a, b]], \tag{3.2}$$

$$E_L(R) = \max[|\log(R(x)/\sqrt{x})|; \quad x \in [a, b]]. \tag{3.3}$$

These best approximations will be abbreviated as C-approx., M-approx. and L-approx. hereafter.

We shall now state two theorems of which Theorem 1 characterizes the best approximations and Theorem 2 clarifies the relation among them.

THEOREM 1. *A function $R \in \mathbf{R}(p, q)$ is the M-approx. (the C-approx.) of the class (p, q) on $[a, b]$, if and only if $r(x) = R(x)/\sqrt{x}$ attains the minimum r' and the maximum r'' alternately at $D(R) + 2$ points of $[a, b]$, and satisfies*

$$r'r'' = 1 \qquad ((r' + r'')/2 = 1). \tag{4}$$

THEOREM 2. *Let R_C, R_M and R_L be the C-approx., the M-approx. and the L-approx., respectively, of the same class on the same interval, then there holds the relation,*

$$R_L = R_M = R_C/\sqrt{(1 - e^2)}, \tag{5}$$

where $e = E_C(R_C)$.

Theorem 1 is a special case of the more general Theorem 4 of [10]. Theorem 2 is given in [9], [10] and partially in [8].

3. Analytic Theory. In this section, we shall show that the M-approx. and the C-approx. of the classes (p, p) and $(p, p - 1)$ can be obtained analytically with the help of elliptic function theory. The Jacobian elliptic function $dn(u, k)$ has the pair of fundamental periods $(2K, 4iK')$, where K and K' are the complete elliptic integrals of the first kind corresponding to the modulus k and the complementary modulus k', respectively, i.e.,

$$k^2 + k'^2 = 1, \qquad K = F(k), \qquad K' = F(k'),$$

$$F(p) = \int_0^{\pi/2} d\theta/\sqrt{(1 - p^2\sin^2\theta)}.$$

Now, letting n be a positive integer, we consider the dn function with the pair of fundamental periods $(2K/n, 4iK')$. The function in question is seen to be of the form $dn(u/M, h)$. The modulus h and the constant M should be so determined that we have

$$H'/H = nK'/K, \qquad M = K/(nH) = K'/H', \tag{6}$$

where

$$h^2 + h'^2 = 1, \qquad H = F(h), \qquad H' = F(h').$$

The modulus h above is determined uniquely, since $F(p')/F(p)$ decreases monotonically from infinity to 0 when p increases from 0 to 1. The function

$$dn(v, h) = dn(u/M, h)$$

thus determined, has, as a function of v, the pair of fundamental periods $(2H, 4iH')$ and therefore, as a function of u, the pair of fundamental periods $(2K/n, 4iK')$.

The transformation $L(n)$: $dn(u, k) \to dn(u/M, h)$ which plays a fundamental role in this paper, is called an *L-transformation* of order n. The L-transformation of order 2 is the familiar *Landen transformation*, and the following formulas are well known:

$$\begin{aligned} dn(u/M, h) &= (k' + dn^2(u, k))/((1 + k')\, dn(u, k)), \\ h &= (1 - k')/(1 + k'), \qquad h' = 2\sqrt{k'}/(1 + k'), \\ M &= 1/(1 + k'). \end{aligned} \tag{7}$$

On the other hand, it is shown in [1] and [13] that the following formulas are valid when n is odd:

$$\begin{aligned} dn(u/M, h) &= dn(u, k) \prod_{m=1}^{[n/2]} \frac{C(2m - 1) + S(2m - 1)\, dn^2(u, k)}{C(2m) + S(2m)\, dn^2(u, k)}, \\ h &= k^n \prod_{m=1}^{[n/2]} S^2(2m - 1), \qquad M = \prod_{m=1}^{[n/2]} (S(2m - 1)/S(2m)), \\ h' &= k'^{2-n} \prod_{m=1}^{[n/2]} D^2(2m - 1), \end{aligned} \tag{8}$$

where

$$\begin{aligned} S(j) &= sn^2(jK/n, k), \\ C(j) &= cn^2(jK/n, k) = 1 - S(j), \\ D(j) &= dn^2(jK/n, k) = 1 - k^2S(j). \end{aligned}$$

We now assert that, with the exception of the last formula for h', the first three formulas of (8) are valid for any value of n. The assertion can be confirmed without any serious difficulties from (7) and (8) by mathematical induction on the largest integer m such that 2^m divides the order n.

Let us now proceed to the derivation of the M-approx. and the C-approx. on an interval $[a, b]$. Putting

$$k = \sqrt{((b-a)/b)}, \qquad k' = \sqrt{(a/b)}, \tag{9}$$

we define the functions R and R^* by the parametric equations on the interval $[0, K]$ of u:

$$x = a/dn^2(u, k), \tag{10.1}$$

$$R(x)/\sqrt{x} = r(x) = dn(u/M, h)/\sqrt{h'}, \tag{10.2}$$

$$R^*(x)/\sqrt{x} = r^*(x) = 2\, dn(u/M, h)/(1 + h'). \tag{10.3}$$

By using (8) and (10), the functions R and R^* are given explicitly as

$$R(x) = \sqrt{(a/h')} \prod_{m=1}^{[n/2]} \frac{C(2m-1)x + S(2m-1)a}{C(2m)x + S(2m)a}, \tag{11.1}$$

$$R^*(x) = \frac{2\sqrt{a}}{1 + h'} \prod_{m=1}^{[n/2]} \frac{C(2m-1)x + S(2m-1)a}{C(2m)x + S(2m)a}. \tag{11.2}$$

Therefore, R and R^* are rational functions. Furthermore, it will be seen that

$$R, R^* \in \mathbf{R}([n/2], [(n-1)/2]), \qquad D(R) = D(R^*) = n - 1.$$

Note here that, when n is even, the denominator of the last factor in each of (11.1) and (11.2) is equal to the constant a.

THEOREM 3. *R and R^* are the M-approx. and the C-approx., respectively, of the class $([n/2], [(n-1)/2])$ on $[a, b]$.*

Proof. Let us examine the behavior of the function $r(x)$ on $[a, b]$. When u varies from 0 to K, $dn(u, k)$ decreases from 1 to k' and hence, x increases from a to b monotonically. On the other hand, it will be observed from the periodic property that $dn(u/M, h)$ attains the maximum 1 and the minimum h' alternately at $n + 1$ points

$$u_j = jK/n, \qquad j = 0, 1, \ldots, n.$$

Therefore, by (10.2), $r(x)$ attains the maximum $r'' = 1/\sqrt{h'}$ and the minimum $r' = \sqrt{h'}$ alternately at $n + 1$ points

$$x_j = a/dn^2(u_j, k), \qquad j = 0, 1, \ldots, n.$$

Since $D(R) + 2 = n + 1$, $r'r'' = 1$, we conclude, from Theorem 1, that R is the M-approx. of the specified class, i.e., of the class (p, p) when $n = 2p + 1$, and of the class $(p, p - 1)$ when $n = 2p$. Quite analogously, it will be shown that R^* is the C-approx. of the same class. This completes the proof.

The maximum relative errors of R and R^* are given by

$$e = E_C(R) = 1/\sqrt{h'} - 1, \tag{12.1}$$

$$e^* = E_C(R^*) = (1 - h')/(1 + h'). \tag{12.2}$$

Another important observation concerning these best approximations is their *symmetry property*:

$$r(y) = r(z), \qquad r^*(y) = r^*(z) \qquad (n\text{: even}), \tag{13.1}$$

(13.2) $$r(y)r(z) = 1 \qquad (n\text{: odd}),$$

whenever $y, z \in [a, b]$ and $yz = ab$. This follows from the property of dn functions [12]:

$$dn(nK - u, k) = dn(u, k) \qquad (n\text{: even}),$$
$$dn(nK - u, k) = k'/dn(u, k) \qquad (n\text{: odd}),$$

since to any such y, z there correspond $v, w \in [0, K]$ such that

$$y = a/dn^2(v, k), \qquad z = a/dn^2(w, k), \qquad v + w = K.$$

The best approximations thus obtained analytically are defined to be the M-approx. and the C-approx., respectively, of order n.

We now turn to the practical determination of the M-approx. of the first five orders on intervals of the form $[a, 1]$, typical values of a being 1/2, 1/4, 1/10, etc. Thus, throughout the rest of this section, k and k' have the values

(14) $$k = \sqrt{(1 - a)}, \qquad k' = \sqrt{a}.$$

Case 1. $n = 1$. This is the trivial case of the constant approximation. From (11.1), (8) and (14), we obtain

(15) $$R(x) = a^{1/4}, \qquad h = \sqrt{(1 - a)}, \qquad h' = \sqrt{a}.$$

Case 2. $n = 2$. This is the case of linear polynomial approximation. From (7), (10) and (14), we obtain

(16) $$\begin{aligned} R(x) &= A_1 x + A_0, \\ A_1 &= 1/\sqrt{(2a^{1/4}(1 + \sqrt{a}))}, \\ A_0 &= \sqrt{a}A_1, \\ h &= (1 - \sqrt{a})/(1 + \sqrt{a}), \\ h' &= 2a^{1/4}/(1 + \sqrt{a}). \end{aligned}$$

In the above two cases, the same results could be obtained by elementary means without resort to the present theory.

Case 3. $n = 3$. From (11.1) and (8), we obtain

$$R(x) = \sqrt{(a/h')}(C(1)x + S(1)a)/(C(2)x + S(2)a),$$
$$h = k^3S^2(1), \qquad h' = D^2(1)/k'.$$

Putting $u = K/3$ in the well-known formulas [12],

(17) $$\begin{aligned} sn(K - u, k) &= cn(u, k)/dn(u, k), \\ cn(K - u, k) &= k'sn(u, k)/dn(u, k), \\ dn(K - u, k) &= k'/dn(u, k), \end{aligned}$$

we have

(18) $$S(2) = C(1)/D(1), \qquad C(2) = aS(1)/D(1).$$

Elementary algebra, using (14) and (18), yields the results:

$$
(19)\qquad
\begin{aligned}
R(x) &= A_0 - B/(x + C),\\
C &= C(1)/S(1),\\
A_0 &= C/a^{1/4},\\
B &= (C^2 - a)/a^{1/4},\\
\sqrt{h} &= (1 - a)^{3/4}S(1),\\
\sqrt{h'} &= D(1)/a^{1/4}.
\end{aligned}
$$

The only remaining task is the computation of

$$
\begin{aligned}
S(1) &= sn^2(K/3, k),\\
C(1) &= cn^2(K/3, k) = 1 - S(1),\\
D(1) &= dn^2(K/3, k) = 1 - (1 - a)S(1).
\end{aligned}
$$

This could be done by an algorithm for Jacobian elliptic functions, but we take another elementary way here. From the first of the duplication formulas [12],

$$
(20)\qquad
\begin{aligned}
sn2u &= 2snucnudnu/(1 - k^2sn^4u),\\
cn2u &= (1 - 2sn^2u + k^2sn^4u)/(1 - k^2sn^4u),\\
dn2u &= (1 - 2k^2sn^2u + k^2sn^4u)/(1 - k^2sn^4u),
\end{aligned}
$$

we have

$$sn(2K/3, k) = 2s\sqrt{((1 - s^2)(1 - k^2s^2))}/(1 - k^2s^4),$$

where s stands for $sn(K/3, k)$. On the other hand, the first equation of (18) is rewritten as

$$sn(2K/3, k) = \sqrt{(1 - s^2)}/\sqrt{(1 - k^2s^2)}.$$

Eliminating $sn(2K/3, k)$ from these equations, we obtain a quartic equation

$$(21)\qquad k^2s^4 - 2k^2s^3 + 2s - 1 = 0,$$

whose unique root in the interval $[1/2, 1]$ is the value of $sn(K/3, k)$. This root can be easily obtained by Newton's method with the initial approximation 1/2.

We notice that the same equation as (21), Eq. (9) of the appendix of [4], appeared in Maehly's analysis, for the present case.

Case 4. $n = 4$. From (11.1) and (8), we obtain

$$R(x) = \frac{(C(1)x + S(1)a)(C(3)x + S(3)a)}{\sqrt{(ah')}(C(2)x + S(2)a)},$$

$$h = k^4S^2(1)S^2(3).$$

Fortunately, there is no computational problem in this case, since sn's and cn's for integral multiples of $K/4$ are known to be expressed in closed forms as functions of k' only [12]. Thus, we have

$$S(1) = (1 - \sqrt{(k'/(1 + k'))})/(1 + \sqrt{k'}),$$
$$S(2) = 1/(1 + k'),$$
$$S(3) = (1 + \sqrt{(k'/(1 + k'))})/(1 + \sqrt{k'}),$$
$$C(1) = \sqrt{k'}(1 + 1/\sqrt{(1 + k')})/(1 + \sqrt{k'}),$$
$$C(2) = k'/(1 + k'),$$
$$C(3) = \sqrt{k'}(1 - 1/\sqrt{(1 + k')})/(1 + \sqrt{k'}).$$

Now it is a simple matter to obtain the following results:

$$\begin{aligned} R(x) &= A_1 x + A_0 - B/(x + C), \\ A_1 &= 1/((4A)^{1/4}(1 + a^{1/4})), \\ A_0 &= (\sqrt{a} + A)A_1, \\ B &= \sqrt{a}AA_1, \\ C &= \sqrt{a}, \\ \sqrt{h} &= (1 - a^{1/4})/(1 + a^{1/4}), \\ \sqrt{h'} &= (4A)^{1/4}/(1 + a^{1/4}), \end{aligned} \tag{22}$$

where $A = 2a^{1/4}(1 + \sqrt{a})$.

Case 5. $n = 5$. From (11.1) and (8), it follows that

$$R(x) = \sqrt{(a/h')}\frac{(C(1)x + S(1)a)(C(3)x + S(3)a)}{(C(2)x + S(2)a)(C(4)x + S(4)a)},$$

$$h = k^5 S^2(1)S^2(3),$$

$$h' = D^2(1)D^2(3)/k'^3.$$

On the other hand, putting $u = K/5, 2K/5$ in (17), we have

$$S(4) = C(1)/D(1), \qquad C(4) = aS(1)/D(1),$$
$$S(3) = C(2)/D(2), \qquad C(3) = aS(2)/D(2),$$
$$D(3) = a/D(2),$$

which, when substituted, yield

$$R(x) = a^{1/4}\frac{(C(1)x + S(1)a)(S(2)x + C(2))}{(C(2)x + S(2)a)(S(1)x + C(1))},$$
$$h = (1 - a)^{5/2}S^2(1)C^2(2)/D^2(2),$$
$$h' = \sqrt{a}D^2(1)/D^2(2).$$

Now we transform R into a continued fraction, and find after some algebraic manipulations the results:

$$
\begin{aligned}
R(x) &= A_0 - B/(x + C - D/(x + E)),\\
A_0 &= a^{1/4}F/G,\\
B &= A_0(F + a/G - G - a/F),\\
E &= aA_0(F/G - G/F)/B,\\
C &= F + a/G - E,\\
D &= CE - aF/G,\\
\sqrt{h} &= (1 - a)^{5/4}S(1)C(2)/D(2),\\
\sqrt{h'} &= a^{1/4}D(1)/D(2),
\end{aligned} \tag{23}
$$

where $F = C(1)/S(1)$, $G = C(2)/S(2)$. Since $C(1) = 1 - S(1)$, $D(1) = 1 - (1 - a)S(1)$, and $S(2)$, $C(2)$ and $D(2)$ can be computed from $S(1)$ by the duplication formulas (20), the only nontrivial task is the computation of $S(1) = sn^2(K/5, k)$. This can be done most efficiently by Salzer's algorithm [11] for Jacobian elliptic functions.

The tables of the M-approxs., given at the end of this paper, were computed on the basis of the above theory. The computations were carried out on the NEAC-2203 computer of Nagoya University, using floating-point double-precision arithmetic with 2-place exponent and 18-place mantissa.

4. Improvement of Newton Iteration. Let us investigate the quality of successive iterates S_i calculated from an arbitrary starting approximation S_0 by the Newton iteration

$$S_{i+1}(x) = N(S_i(x)) = (S_i(x) + x/S_i(x))/2$$

on an interval $[a, b]$. By defining $s_i(x)$ as $s_i(x) = S_i(x)/\sqrt{x}$, it follows that

$$s_{i+1}(x) = N^*(s_i(x)) = (s_i(x) + 1/s_i(x))/2.$$

The function N^* defined by

$$N^*(y) = (y + 1/y)/2 \qquad (y > 0)$$

has the properties:

$$
\begin{gathered}
N^*(y) \geqq N^*(1) = 1,\\
yz = 1 \to N^*(y) = N^*(z),\\
(yz - 1)(y - z) > 0 \rightleftarrows N^*(y) > N^*(z).
\end{gathered} \tag{24}
$$

Here, for the sake of simplicity, we agree that any r and s symbols with prime or double primes denote the minimum or the maximum, respectively, of the corresponding function.

Now, it follows, from (24), that

$$s_i' \geqq 1, \qquad i = 1, 2, \ldots,$$

where the equality holds only if

$$s_0' \leqq 1 \leqq s_0''.$$

At any rate, we have

$$s_i' s_i'' > 1, \qquad i = 1, 2, \ldots.$$

This inequality means, in view of Theorem 1, that every iterate S_i is biassed upward, and is not satisfactory for subsequent iterates. A remedy for this drawback of the conventional Newton iteration is to readjust every iterate by a correcting factor so that the above mentioned inequality may be turned into an equality.

The *improved Newton iteration* which incorporates the readjustment into the conventional one is defined by

$$\begin{aligned} R_{i+1}(x) &= C_{i+1} N(R_i(x)), \\ R_0(x) &= C_0 S_0(x). \end{aligned} \tag{25}$$

From (24), the correcting factors are given by

$$\begin{aligned} C_0 &= 1/\sqrt{(s_0' s_0'')}, \\ C_i &= 1/\sqrt{N^*(r_{i-1}'')}, \qquad i = 1, 2, \ldots, \end{aligned} \tag{26}$$

where $r_i(x) = R_i(x)/\sqrt{x}$.

Let us compare the convergence rates of the conventional and the improved Newton iterations. We may assume $C_0 = 1$ for simplicity, since, otherwise, the situation would be more favorable for the improved version. Letting e_i and f_i denote the maximum relative errors of R_i and S_i, respectively, we have

$$\begin{aligned} e_i &= r_i'' - 1, \\ f_i &= s_i'' - 1, \\ e_{i+1} &= \sqrt{(1 + e_i^2/(2(1 + e_i)))} - 1, \\ f_{i+1} &= f_i^2/(2(1 + f_i)). \end{aligned} \tag{27}$$

Starting with $e_0 = f_0$, we obtain

$$e_1 = \sqrt{(1 + f_1)} - 1 \leqq f_1/2.$$

Since it is difficult to obtain similar relations between e_i and f_i exactly, we content ourselves with approximate relations. Thus, applying the approximate recurrence formulas

$$e_{i+1} \sim e_i^2/4, \qquad f_{i+1} \sim f_i^2/2,$$

instead of the exact ones of (27), we obtain

$$e_2 \sim f_2/8, \qquad e_3 \sim f_3/128, \ldots,$$

and, in general, $e_i \sim f_i/2^{2^i - 1}$. This shows a remarkable acceleration of convergence accomplished by the use of the improved Newton iteration. Incidentally, it is an open question to prove or disprove the inequality

$$e_i < f_i/2^{2^i - 1}.$$

The excellence of the improved Newton iteration illustrated above suggests the natural question: What is the result when it is applied to the M-approx. of order n? The following theorem is a very interesting answer to this question.

THEOREM 4. *If R_0 is the M-approx. of order n on an interval $[a, b]$, then R_i obtained from R_0 by the improved Newton iteration is the M-approx. of order $2^i n$ on $[a, b]$.*

Proof. As shown previously in Section 3, R_0 is given by

$$\begin{aligned} x &= a/dn^2(u, k), \\ R_0(x)/\sqrt{x} &= r_0(x) = dn(u/M_0, h_0)/\sqrt{h'_0} \end{aligned} \tag{28}$$

on the interval $[0, K]$ of u, where

$$\begin{aligned} k &= \sqrt{((b - a)/b)}, \\ dn(u/M_0, h_0) &= L(n) \cdot dn(u, k). \end{aligned}$$

If we assume that the theorem is true for $i = j$, then we have

$$\begin{aligned} R_j(x)/\sqrt{x} &= r_j(x) = dn(u/M_j, h_j)/\sqrt{h'_j}, \\ dn(u/M_j, h_j) &= L(2^j n) \cdot dn(u, k). \end{aligned} \tag{29}$$

Since $r'_j = \sqrt{h'_j}$, $r''_j = 1/\sqrt{h'_j}$, $r'_j r''_j = 1$, it follows, from (24) and (26), that

$$C_{j+1} = 1/\sqrt{N^*(\sqrt{h'_j})} = \sqrt{(2\sqrt{h'_j}/(1 + h'_j))}.$$

When use is made of this value of C_{j+1} in the equation

$$r_{j+1}(x) = C_{j+1} N^*(r_j(x)),$$

which is a direct consequence of (25) for $i = j$, we obtain

$$r_{j+1}(x) = (h'_j + dn^2(u/M_j, h_j))/(C_{j+1}(1 + h'_j) dn(u/M_j, h_j)).$$

Comparing this with Eqs. (7), we find that

$$\begin{aligned} C_{j+1} &= \sqrt{h'_{j+1}}, \\ r_{j+1}(x) &= dn(u/M_{j+1}, h_{j+1})/\sqrt{h'_{j+1}}, \\ h_{j+1} &= (1 - h'_j)/(1 + h'_j), \\ h'_{j+1} &= 2\sqrt{h'_j}/(1 + h'_j), \\ M_{j+1} &= M_j/(1 + h'_j), \\ dn(u/M_{j+1}, h_{j+1}) &= L(2) \cdot dn(u/M_j, h_j). \end{aligned} \tag{30}$$

The last equation of (30), when combined with the second one of (29), yields

$$dn(u/M_{j+1}, h_{j+1}) = L(2^{j+1} n) \cdot dn(u, k),$$

since, in general, the composition of $L(p)$ and $L(q)$ is $L(pq)$. Thus, the theorem is proved for $i = j + 1$, and, therefore, for every i by induction.

In order to carry out the improved Newton iteration described in Theorem 4, the correcting factors C_i should be computed in advance. The following recurrence formula

(31) $$C_{i+1} = \sqrt{(2C_i/(1 + C_i^2))}, \qquad C_0 = \sqrt{h_0'} = 1/(1 + e_0),$$

which is easily established from (30), serves for this purpose.

If, as is often the case with the computation by a computer, the number of iterations is prescribed, then it is desirable to obtain a C-approx. rather than an M-approx. in the last step. Fortunately, this is accomplished without altering the algorithm largely. It is sufficient to replace only the last step, say, the mth step, with

$$R_m^*(x) = C_m^* N(R_{m-1}(x)),$$

where C_m^* is given, from (10.3) and (30), by

(32) $$C_m^* = 2C_m^2/(1 + C_m^2).$$

$R_m^*(x)$, thus computed, is the C-approx. of order $2^m n$ and its maximum relative error is given by

$$e_m^* = 1 - C_m^*.$$

Let us now summarize the computing procedure into an algorithm.

Algorithm. Improved Newton iteration.

Preparation. Determine the number of iterations m, and the order n of the M-approx. used for a starting value. Compute $C_1/2, C_2/2, \ldots, C_{m-1}/2$, and $C_m^*/2$ by (31) and (32).

First Step. Compute the starting value $R_0(x)$.

Loop. For $1 \leqq i \leqq m - 1$, iterate

$$R_i(x) = (C_i/2)(R_{i-1}(x) + x/R_{i-1}(x)).$$

Last Step. Compute $R_m^*(x)$ by

$$R_m^*(x) = (C_m^*/2)(R_{m-1}(x) + x/R_{m-1}(x)).$$

5. Practical Considerations. The improved Newton iteration discussed in the last section is an excellent computing procedure. For one iteration, it requires the same amount of computational effort, an addition, a multiplication and a division, and is nevertheless more than two times as accurate as the conventional one. The only conceivable disadvantage is that the multiplications of the factors $C_i/2$ are a little slower than those of the factor 1/2 in binary computers.

The important problem which is left untouched is the choice of the order of the M-approx. to be used as the starting approximation. The number and the kinds of arithmetic operations required for computing an M-approx. of the order n expressed in a continued fraction are shown below, where A stands for addition-subtraction, M for multiplication and D for division.

	A	M	D
n: odd	$n-1$	0	$(n-1)/2$
n: even	$n-1$	1	$n/2-1$

In view of the fact that the result obtained from an M-approx. of order n after one iteration is identical with an M-approx. of order $2n$ and is more accurate than an M-approx. of order $2n - 1$, we compare the amounts of computation required for the three cases. The results of the comparisons are shown in the following table.

	A	M	D
M_1	0	0	0
M_2	1	1	0
$C \times N(M_1)$	1	1	1

	A	M	D
M_3	2	0	1
M_4	3	1	1
$C \times N(M_2)$	2	2	1

	A	M	D
M_5	4	0	2
M_6	5	1	2
$C \times N(M_3)$	3	1	2

	A	M	D
M_7	6	0	3
M_8	7	1	3
$C \times N(M_4)$	4	2	2

Inspecting the table we find that the choices $n = 1$ and $n \geqq 6$ should be excluded from consideration, since the same or even better approximations could be obtained by other choices with less computational effort. It is interesting to note that the choice $n = 4$ is unexpectedly superior to that of $n = 2$, at least in the count of multiplications and divisions. We can say hardly anything further, however, for the remaining four choices. There are many other conditions to be taken into account which cannot be discussed here in general.

Table of M-approximations on $[a, 1]$

$n = 2, \quad R(x) = A_1 x + A_0$

	$a = 1/2$	$a = 1/4$	$a = 1/16$
A_1	0.59017 85321	0.68658 90480	0.89442 71910
A_0	0.41731 92422	0.34329 45240	0.22360 67977
e_0	0.00749 77743	0.02988 35720	0.11803 39887

	$a = 1/\sqrt{10}$	$a = 1/10$	$a = 1/100$
A_1	0.65327 65093	0.82190 09419	1.19891 57337
A_0	0.36736 43780	0.25990 78987	0.11989 15734
e_0	0.02064 08873	0.08180 88406	0.31880 73070

$n = 3, \quad R(x) = A_0 - B/(x + C)$

	$a = 1/2$	$a = 1/4$	$a = 1/16$
A_0	2.54163 91882	2.18518 30604	1.68212 58623
B	4.83752 82229	3.02289 91727	1.28977 37082
C	2.13725 52822	1.54515 77602	0.84106 29311
e_0	0.00032 28502	0.00252 93327	0.01877 95823

	$a = 1/\sqrt{10}$	$a = 1/10$	$a = 1/100$
A_0	2.29635 64606	1.82780 35809	1.27346 54826
B	3.53268 53812	1.70087 90336	0.48120 83248
C	1.72202 44124	1.02784 94879	0.40270 51447
e_0	0.00146 11372	0.01107 77906	0.07479 71524

$n = 4, \quad R(x) = A_1 x + A_0 - B/(x + C)$

	$a = 1/2$	$a = 1/4$	$a = 1/16$
A_1	0.29508 51497 68	0.34322 01291 85	0.44582 68699 84
A_0	1.05584 61593 35	0.89968 99069 52	0.66874 03049 76
B	0.59905 34042 71	0.36403 99211 80	0.13932 08968 70
C	0.70710 67811 87	0.50000 00000 00	0.25000 00000 00
e_0	0.00001 39494 67	0.00021 67553 51	0.00311 04574 65
	$a = 1/\sqrt{10}$	$a = 1/10$	$a = 1/100$
A_1	0.32660 41728 27	0.41031 63482 08	0.58823 15087 81
A_0	0.94895 58546 26	0.73716 05228 76	0.46805 64499 08
B	0.43035 57848 93	0.19207 89905 24	0.04092 33299 03
C	0.56234 13251 90	0.31622 77660 17	0.10000 00000 00
e_0	0.00010 43520 80	0.00154 54483 60	0.01908 49315 04

$n = 5, \quad R(x) = A_0 - B/(x + C - D/(x + E))$

	$a = 1/2$	$a = 1/4$	$a = 1/16$
A_0	4.23606 54239 70	3.64197 75773 02	2.80378 55369 82
B	24.27865 63619 77	15.34365 23620 24	6.85055 23463 25
C	6.72879 05912 69	4.93393 81498 86	2.83472 72475 43
D	0.32178 82625 97	0.16349 79998 78	0.04321 24689 67
E	0.42215 13205 04	0.29411 22236 87	0.13887 95558 48
e_0	0.00000 06028 07	0.00001 85948 42	0.00051 85424 23
	$a = 1/\sqrt{10}$	$a = 1/10$	$a = 1/100$
A_0	3.82726 28073 50	3.04643 17009 11	2.12519 04940 28
B	17.84758 11560 46	8.86587 47888 42	2.84184 47997 55
C	5.46681 44323 38	3.38925 64040 23	1.52183 61874 77
D	0.20546 54759 00	0.06766 96256 53	0.00758 48567 14
E	0.33280 96905 39	0.17980 64913 92	0.04914 41073 35
e_0	0.00000 74573 53	0.00021 64864 94	0.00496 90136 43

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